THE MANIFOLD APPLICATIONS OF PASCAL'S TRIANGLE

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I. INTRODUCTION

The purpose of this paper is to examine the uses of Pascal's triangle and to show how it relates to many different fields of mathematics. The great apparent simplicity of Pascal's triangle has prompted some people, including Kepler and others, to philosophize about the aesthetic value of such aspects of mathematics. Morris Kline stated:

> Perhaps the best reason for regarding mathematics as an art is not so much that it affords an outlet for creative activity as that it provides spiritual values. It puts man in touch with the highest aspirations and loftiest goals. It offers intellectual delight and the exaltation of resolving the mysteries of the universe.[1]

Certainly it is very interesting that such a wide range of applications can result out of a single array of numbers.

II. HISTORY OF PASCAL'S TRIANGLE

Blaise Pascal (1623-1662) was one of the most brilliant mathematicians of the seventeenth century until he gave up mathematics for mysticism. A child prodigy, he was only thirteen when he discovered the sequence of sequences known as Pascal's triangle.[2] Actually, it was previously given by Omar Khayyém, and it figured in the <u>Precious Mirror of the Four Elements</u>,[3] published in 1303 by the Chinese mathematician Chu Shi Hei,[4] who lived at the time the Chinese Empire was sprawling into Europe. (Many other mathematical advances, such as the Pythagorean Theorem, were actually the result of Eastern culture and rediscovered much later in the West.)

Pascal's Triangle is an orderly way of arranging the coefficients of the binomial expansion of $(x + y)^n$ where n is a nonnegative integer. The binomial theorem states that:

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \tag{1}$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{2}$$

It follows from this that

$$\binom{n}{r} = \binom{n}{n-r} \tag{3}$$

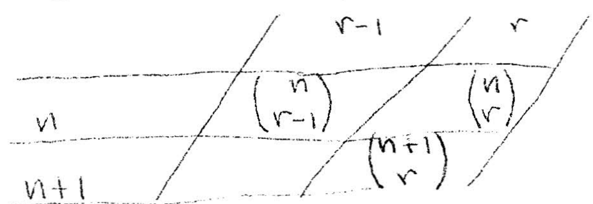
and

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r} \tag{4}$$

These formulas suggest an orderly way of arranging the terms for various values of n and r. This is known as Pascal's Triangle:

		r=0	r=1	r=2	r=3	r=4	r=5
n = 0		1					
n = 1		1	1				
n = 2		1	2	1			
n = 3		1	3	3	1		
n = 4		1	4	6	4	1	
n = 5		1	5	10	10	5	1

$\binom{n}{r}$ can be found by lining up the nth row, starting with n = 0, and the rth right diagonal, starting with r = 0. Therefore, it follows from equation (4) that each number is the sum of the two integers immediately above it, as follows:



For small values of n, Pascal's Triangle provides an easy way to find the coefficients of the binomial expansion of $(x + y)^n$

since these constitute the nth row of Pascal's Triangle.

III. COMBINATORIAL ANALYSIS

The number of distinguishable combinations of n things taken r at a time without repetitions is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad (2)$$

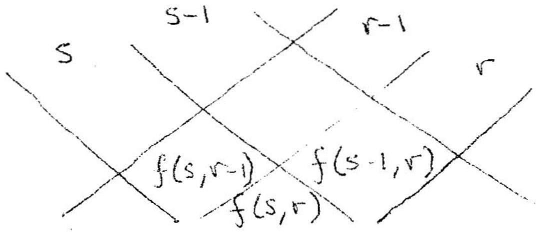
These are, in fact, the same numbers as those found in Pascal's Triangle! (It is not a mere coincidence, however, as the binomial theorem follows as a direct result of this.)

The number of distinguishable combinations of s things taken r at a time *with* repetitions is also found in Pascal's Triangle; let is be $f(s,r)$ since it is a function of two independent variables. If any one of the s things is arbitrarily labeled X, then $f(s,r)$ can be divided into two subsets: the number of combinations that include X at least once, and the number of combinations that do not include X at all.

For the number of combinations that do not include X, there are $(s - 1)$ things taken r at a time with repetitions, and therefore the number is $f(s - 1, r)$. For the number of combinations that include X at least once, there are $(r - 1)$ other choices from s things (since repetitions are allowed), so the number is $f(s, r - 1)$. Therefore:

$$f(s, r) = f(s - 1, r) + f(s, r - 1) \qquad (5)$$

This is a true recursive definition because $f(s, r)$ can be found in terms of numbers of the form $f(s, 0)$ and $f(1, r)$, which are both equal to one ($f(s, 0)$ is the empty set). This is very similar to Pascal's Law, Equation (4), and can be used as the basis for Pascal's Triangle, if s is the number of left diagonals starting at s = 1, and if r is the number of right diagonals starting at r = 0. Note that r is the same for both triangles.



The border is the same because $f(s, 0) = f(1, r) = \binom{0}{0} = \binom{n}{n} = 1$.

By the geometry of Pascal's Triangle it is easily proven that:

$$n = s + r - 1 \tag{6}$$

Therefore, since $f(s, r)$ is identical to $\binom{n}{r}$,

$$f(s, r) = \binom{n}{r} = \binom{s + r - 1}{r} = \frac{(s + r - 1)!}{r!(s-1)!} \tag{7}$$

Equation (7) is very useful in solving many types of problems, such as:

> A store stocks s kinds of candy bars. How many distinguishable combinations are there of buying r candy bars?
>
> How many possible results are there of rolling r s-sided dice, if the dice are not distinguishable?
>
> How many ways are there of putting r identical balls into s different cells?

One way of proving Equation (7) is by comparing the last example to the number of permutations of the possible linear arrangements of (s - 1) bars and r stars (e.g., */**//*/***///**)

This number is:

$$f(s, r) = \binom{s + r - 1}{r} \tag{7}$$

Because s and r are not reversible (i.e., $f(s, r) \neq f(r, s)$), some of the symmetry caused by Equation (3) is broken. It follows from Equations (3) and (6) that:

$$f(s, r) = f(r + 1, s - 1) \tag{8}$$

If, in the third example, it is required that there be at least one ball in each cell, then the number of combinations would be:

$$f(s,\ r - s) = \binom{r-1}{s-1} \qquad (9)$$

It follows that the number of permutable partitions of a number r is:

$$\sum_{s=1}^{r} \binom{r-1}{s-1} = \sum_{s-1=0}^{r-1} \binom{r-1}{s-1} 1^{r-s} 1^{s-1} = (1+1)^{r-1} = 2^{r-1} \qquad (10)$$

This is the sum of the numbers in the (r - 1)th row (which are also the coefficients of the (r - 1)th binomial expansion).

Of course, the number of permutations of s things taken r at a time with repetitions is:

$$P = s^r \qquad (11)$$

Equation (11) is more useful in probability theory than Equation (7), but Equation (7) is useful in a wide range of problems that might otherwise be very difficult. It is important to note that Pascal's Triangle was very useful in the derivation of Equation (7), but it was not necessary for its proof.

IV. OTHER PROPERTIES OF PASCAL'S TRIANGLE

Since the sum of the numbers in any diagonal is found in Pascal's triangle by:

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

numbers of the form $\binom{n}{1}$ are consecutive integers, numbers of the form $\binom{n}{2}$ are triangular numbers (if arranged like bowling pins, it is easy to see that it is the sum of consecutive integers), and numbers of the form $\binom{n}{3}$ are tetrahedral numbers (if equilateral triangles of cannon balls, with consecutive integers for the number of balls to a side, are stacked up, the result is a tetrahedron). Pyramidal numbers (the sum of consecutive squares, geometrically a square-based pyramid) can be found by adding two consecutive tetrahedral numbers[5] or by adding a triangular number to twice the previous tetrahedral number.[6] The sum of consecutive cubes is found to be the square of a triangular number. Algebraically,

$$\sum_{i=1}^{n} i^2 = \binom{n+1}{3} + \binom{n+2}{3} = \binom{n+1}{2} + 2\binom{n+1}{3} = \frac{n(n+1)(2n+1)}{6}$$

and $$\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2 = \frac{n^2(n+1)^2}{4}$$

A three-dimensional version of Pascal's triangle can be designed by using the trinomial expansion and arranging the coefficients in a tetrahedron so that Pascal's triangle is a special case. This is useful in probability theory, but as far as I know its properties have not been as fully investigated as have been Pascal's triangle. The multinomial expansion[7] is also important in probability but it is impossible to construct the coefficients geometrically in three-dimensional space. The following result is just one of countless other uses of the binomial coefficients of which I haven't the space to put here.

$$n! = \sum_{r=0}^{n} (-1)^r \binom{n}{r} (n-r)^n$$

V. THE FIBONACCI SEQUENCE

Pascal's Triangle can be arranged as follows:

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

The numbers connected by lines are called the ascending diagonals. If u_n is the sum of the terms in the nth ascending diagonal, then it follows from Equation (4) that:

$$u_n = u_{n-1} + u_{n-2}$$

Since $u_1 = u_2 = 1$, this forms a recurrent sequence of the second order:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

This is known as the Fibonacci Sequence, named after one of the very few great methematicians of the Middle Ages, Fibonacci, also known as Leonardo of Pisa. In 1202 Fibonacci (a contraction of filius Bonacci, i.e., son of Bonacci) wrote a very important book, Liber Abaci ("A book about the abacus"). On pages 123-124 of the 1228 edition, Fibonacci esplains his famous rabbit problem, the solution of which is the Fibonacci sequence:

> A pair of rabbits is placed in a walled enclosure to find out how many offspring this pair will produce in the course of a year if each pair of rabbits gives birth to a new pair each month starting with the second month of its life. Since the first pair has offspring in the first month, double the number, and in this month there are two pairs. Of these, one pair, the first, gives birth in the following month as well, so that in the second month there are three pairs. Of these, two pairs have offspring in the following month, so that in the third month two additional pairs of rabbits are born, and the total number of pairs of rabbits in this month reaches five. Three of these five pairs have offspring that month, and the number of pairs reaches eight in the fourth month. Five of these pairs produce another five pairs, which, together with the eight pairs already in existence, make 13 pairs in the fifth month. Five of these 13 pairs have no offspring that month, while the remaining eight pairs do give birth, so that in the sixth month there are 21 pairs. Adding to these the 13 pairs born in the seventh month, we have a total of 34 pairs. Adding to these the 21 pairs born in the eighth month, we have a total of 55 pairs. Adding to these the 34 pairs born in the ninth month, we have a total of 89 pairs. Adding to these the 55 pairs born in the tenth month, we have a total of 144 pairs. Adding to these the 89 pairs born in the eleventh month, we have a total of

233 pairs. Finally, adding to these the 144 pairs born in the final month, we have a total of 377 pairs. This is the number of pairs produced by the first pair in the given place at the end of the year.

Pairs:	1
first month:	2
second month:	3
third month:	5
fourth month:	8
fifth month:	13
sixth month:	21
seventh month:	34
eighth month:	55
ninth month:	89
tenth month:	144
eleventh month:	233
twelfth month:	377

By examining the above table you can see how we arrive at the result; that is, we add the first number to the second, that is, one to two; the second to the third; the third to the fourth; the fourth to the fifth; and so forth, until we add the tenth and eleventh numbers, 144 and 233, and thus obtain the total number of rabbits in question, 377.

The Fibonacci Sequence has many unusual properties, such as the property that the partial sum of the first n Fibonacci numbers is $(u_{n+2} - 1)$. This sequence of partial sums is also the sequence of the sums of the terms in the ascending diagonals of Pascal's Triangle if the first right diagonal (a column in the last figure) is deleted. If the first two right diagonals are deleted, the sums of the terms in the ascending diagonals are the partial sums of the previous partial sums. In general, if the first k terms of an ascending diagonal are omitted, the sum of the remaining terms is a kth order partial sum of the Fibonacci Sequence.[8] Thus, there is a very close relationship between Pascal's Triangle and the Fibonacci Sequence.

The relation

$$u_n^2 - u_{n-1}\, u_{n+1} = (-1)^n$$

was implied by Kepler, who was familiar with the Fibonacci sequence when he said:

> For we will always have as 5 is to 8 so is 8 to 13, practically, and as 8 is to 13, so is 13 to 21 almost. I think that the seminal faculty is developed in a way analogous to this proportion which perpetuates itself, and so in the flower is displayed a pentagonal standard, so to speak. I let pass all other considerations which might be adduced by the most delightful study to establish this truth.[9]

In 1879, Catalan showed that[10]

$$u_{n+1-p}\, u_{n+1+p} - u_{n+1}^2 = (-1)^{n+2-p}\, u_p$$

and a few years later he gave the following result:[11]

$$2^{n-1} u_n = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} 5^i \binom{n}{2i+1}$$

$$= \binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + 5^3\binom{n}{7} + \ldots$$

Once again, we find that the Fibonacci Sequence can be found in terms of the numbers in Pascal's Triangle in a way entirely different from the way previously mentioned. There are many other general formulas relating different Fibonacci numbers, ranging from Pythagorean triples to number theory, many of which are most easily proved by induction.

Professor Jekuthiel Ginsburg showed that the reciprocal of the eleventh term of the Fibonacci Sequence contains the entire Fibonacci Sequence:[12]

$$\frac{1}{u_{11}} = \frac{1}{89} = \sum_{i=1}^{\infty} u_i\, 10^{-i-1}$$

However, this is a coincidence to the extent that if we were not on a system of base ten, a similar series would only converge to the reciprocal of a Fibonacci number if we used base two, three, or eight.

Most of the uses of Fibonacci numbers aren't very practical but some of the following have been used to calculate pi:[13]

$$\operatorname{arccot} u_n/u_{n+1} \; - \; \operatorname{arccot} u_{n-1}/u_n \; = \; (-1)^n \operatorname{arccot} u_{2n}$$

$$\operatorname{arccot} u_{2p-1} \; + \; \operatorname{arccot} u_{2p+2} \; = \; \operatorname{arccot} u_{2p}/u_3$$

$$\operatorname{arccot} u_{2p-1} \; + \; \operatorname{arccot} u_{2p+3} \; = \; \operatorname{arccot} u_{2p+1}/u_4$$

VI. THE GOLDEN SECTION

The ratio of two consecutive Fibonacci numbers, u_{n+1}/u_n, approaches $\frac{1}{2}(1 + \sqrt{5})$ as $\underline{n}$ increases. This number is so significant that it is called the golden ratio and has the symbol ϕ (phi). It can be defined as

$$\phi = \frac{AB}{BC}, \quad \text{if} \quad \frac{AB}{BC} = \frac{AC}{AB},$$

for the line segment ABC. Solving the resulting quadratic gives $\phi = 1.61803398\cdots$ as its positive root. The ancient Greeks used the golden rectangle (one where phi is the ratio between two adjacent sides) in their art and architecture (the front of the Parthenon is a golden rectangle) because it was thought to be the rectangle most pleasing to the eye. Kepler, a "confirmed mystic,"[14] called phi "the divine proportion" because it occurs so frequently in mathematics. Kepler said:

> Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel.[15]

Phi is so common in mathematics partially because it is the positive root of $x^2 - x - 1 = 0$, and is equal to $2 \cos \pi/5$. But it also occurs where you might not expect, such as in phyllotaxis, which describes the arrangement of leaves on the stems of plants. The fractional rotation between two successive leaves is the ratio of alternate Fibonacci numbers (consecutive if you go around the stem the long way). The ratio is 1/2 for the basswood, Indian corn, and all the grasses; 1/3 for sedges; 2/5 for the apple, cherry, and most of our common shrubs; 3/8 for the common plantain; and 5/13 for the common house leek. Other ratios are found in the pine family and in many small plants.[16]

H. E. Licks claimed that these same ratios are related to astronomy in the following way:

> The furthermost planet from the sun is Neptune, then follow Uranus, Saturn, Jupiter, the Asteroids, and Mars, then the Earth, Venus, and Mercury. Neptune makes its revolution around the sun in about 60,000 days; Uranus in 30,000days or about 1/2 the time of Neptune; in like manner Saturn's period is nearly 1/3 of that of Uranus, Jupiter's period 2/5 that of Saturn, and so on until we come to the Earth, following closely the same series as given above for the leaves on a stem. Thus the mathematical expression of the arrangement of the leaves of plants is approximately the same as that of the periods of the exterior planets. These arrangements of leaves ensure to plants a better distribution of the light and heat of the sun; the periods of the planets render them stable under the laws of gravitation. Perhaps the botanist, had he known that these figures apply both to leaves and planets, might have foretold the discovery of the Asteroids or announced the existence of Neptune.[17]

Phi, like pi, occurs as the limit of certain series which have no apparent relation to phi, such as[18]

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}$$

and

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

The latter is a continued fraction, which is any fraction of that form. All rational numbers terminate when converted to a continued fraction because of the Euclidean algorithm. Many irrational numbers form simple patterns when converted to a continued fraction. The convergents (rational approximations) of phi are numbers of the form u_{n+1}/u_n .

To find u_n in terms of n, Binet showed that:[19]

$$u_n = \frac{\phi^n - (-\phi)^{-n}}{5}$$

This formula is useful in proving many properties of Fibonacci numbers. To find u_n in terms of u_{n-1} ,

$$u_n = \tfrac{1}{2}u_{n-1} + \tfrac{1}{2}\sqrt{5u_{n-1}^{\,2} + 4(-1)^{n-1}}$$

or, to the nearest integer,

$$u_n = \phi\, u_{n-1} .$$

In geometry, phi occurs frequently. It is closely associated with the logarithmic spiral; the regular decagon, pentagon, and pentagram; and three of the five Platonic solids:[20]

> The Icosahedron. The twelve vertices of a regular icosahedron are divisible into three coplanar groups of four. These lie at the corners of three golden rectangles which are symmetrically situated with respect to each other, being mutually perpendicular, their one common point being the centroid of the icosahedron.
>
> The Octahedron. An icosahedron can be inscribed in an octahedron so that each vertex of the former divides an edge of the latter in the golden section.
>
> The Dodecahedron. The centroids of the twelve pentagonal faces of a dodecahedron are divisible into three coplanar groups of four. These quadrads lie at the corners of three mutually perpendicular, symmetrically placed golden rectangles, their one common point being the centroid of the dodecahedron.

Phi is so common that mathematicians continue to find it in new places; it can even be found in the 3-4-5 triangle, the Cross of Lorraine, and a sunflower!

VII. THE CALCULUS OF FINITE DIFFERENCES

The numbers in Pascal's Triangle (sometimes called figurate numbers) have another practical use in the calculus of finite differences, the branch of mathematics "which deals with the successive differences of the terms in a sequence of numbers."[21] It is useful when a table is available for a function of x, f(x), when the arguments x_i are equi-spaced. The advancing differences are defined as:

$$\Delta^0 f(x_i) = f(x_i)$$
$$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$$
$$\Delta^n f(x_i) = \Delta^{n-1} f(x_{i+1}) - \Delta^{n-1} f(x_i)$$
$$= \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(x_{i+n-r})$$

If u_i is defined as

$$u_i = \frac{x - a}{h}$$

where $h = \Delta x$, and a is any arbitrary origin, then Newton's formula gives:

$$f(u) = \sum_{r=0}^{\infty} \binom{u}{r} \Delta^r f(0)$$
$$= f(0) + u \Delta f(0) + \frac{u(u-1)}{2!} + \cdots \quad (1)$$

This is an extremely important formula, because it allows us to construct a formula for many sequences that follow simple laws. For example, if we want to find a formula for the sum of the first n squares, we need only calculate a few differences, set h = 1 and a = 0 for convenience, and we obtain a formula for the sum in terms of _n_ because the series terminates rapidly. In fact, if $f(x)$ is a polynomial in _x_, where the highest degree of _x_ is _n_, then the series terminates after $(n + 1)$ terms because $\Delta^{n+1} f(x) = 0$. If $f(x)$ is not a polynomial in _x_, then Newton's formula can be used for interpolation if at some point the differences are regarded as negligible.

Newton's formula can also be used to approximate integrals. Since $f(x)$ can be found in terms of $y_0, y_1, y_2, \ldots, y_k$ and _a_, _h_, and _x_, the result can be integrated between the limits of _a_ and $(a + kh)$. In this way the trapezoidal rule can be derived for $k = 1$ and Simpson's rule for $k = 2$:

$$k = 1: \quad A = \tfrac{1}{2}h(y_0 + y_1)$$

$$k = 2: \quad A = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$k = 3: \quad A = \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

$$k = 4: \quad A = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

The approximation for the area, A, gets progressively more accurate as _k_ increases and if $f(x)$ is a polynomial in _x_ where the highest degree of _x_ is less than or equal to _k_ then the area is exact and yields the same result as integration . Of course, for any set of ordinates $y_0, y_1, y_2, \ldots, y_i$, the formula can be repeated if _i_ is a multiple of _k_. This produces the trapezoidal rule and Simpson's rule in their usual form as well as more accurate formulas for the area under a curve if _k_ has a larger value, e.g.:

$$A = \frac{3}{8}h(y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + \cdots + y_{3i})$$

It should be noted that in Newton's formula u need not be an integer since $\binom{u}{r}$ can be defined as

$$\binom{u}{r} = \frac{u(u-1)(u-2)\cdots(u-r+1)}{1\cdot 2\cdot 3\cdot\cdots\cdot r}$$

In some old books, the notation $\frac{u^{(r)}}{r!}$ is sometimes used for $\binom{u}{r}$. This draws closer the similarity between Newton's formula and the Maclaurin series,

$$f(u) = f(0) + u\,f'(0) + \frac{u^2}{2} f''(0) + \cdots \tag{2}$$

More generally,

$$f(x) = f(a) + \binom{\frac{x-a}{h}}{1} \Delta f(a) + \binom{\frac{x-a}{h}}{2} \Delta^2 f(a) + \cdots \tag{3}$$

and

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots \tag{4}$$

In order to generalize the two formulas, I will define the operation Δ_h^n such that

$$\Delta_h^1 f(x) = \Delta_h f(x) = \lim \frac{f(x+h)-f(x)}{h}$$

and

$$\Delta_h^n f(x) = \Delta_h\left(\Delta_h^{n-1} f(x)\right)$$

where h is allowed to have any real value. This operation is a generalized derivative since $\Delta_h^n f(x) = f^{(n)}(x)$ if $h = 0$. If h is not equal to zero,

$$\Delta_h^n f(x) = \frac{\Delta^n f(x)}{h^n}$$

The Taylor series (the generalized Maclaurin series) is possible because the derivatives of $\frac{(x-a)^r}{r!}$ form a simple sequence and are all equal to zero at $x = a$ except for one which is equal to one. A similar reasoning holds for Newton's formula since $\Delta^n \binom{u}{r} = \binom{u}{r-n}$, which follows from Pascal's Law. In order to construct a general series using Δ_h^n, it is necessary to find a function of x, a, h, and r with a similar property as mentioned before with respect to Δ_h^n. The only one is

$$\binom{x-a}{r}_h = \frac{(x-a)(x-a-h)(x-a-2h)\cdots(x-a-rh+h)}{r!}$$

since $\Delta_h^n \binom{x-a}{r}_h = \binom{x-a}{r-n}_h$, which generalizes Pascal's Law.

Therefore it follows that

$$f(x) = f(a) + \binom{x-a}{1}_h \Delta_h f(a) + \binom{x-a}{2}_h \Delta_h^2 f(a) + \cdots \quad (5)$$

Note that Newton's formula is a special case of this where a = 0 and h = 1; and the Taylor series is a special case where h = 0. Thus I have succeeded in unifying the two formulas.

By repeated integration by parts of $\int_a^x f(t)dt$, we can derive the infinite series

$$\int_a^x f(t)dt = (x-a)f(x) - \frac{(x-a)^2}{2!}f'(x) + \frac{(x-a)^3}{3!}f''(x) - \cdots \quad (6)$$

or, with the remainder term,

$$\int_a^x f(t)dt = \sum_{i=0}^{n} (-1)^i \frac{(x-a)^{i+1}}{(i+1)!} f^{(i)}(x) + (-1)^{n+1} \int_a^x \frac{(t-a)^{n+1}}{(n+1)!} f^{(n+1)}(t)\,dt \quad (7)$$

It follows from this that

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(x) - \frac{(x-a)^2}{2!}f''(x) + \frac{(x-a)^3}{3!}f'''(x) - \cdots \quad (8)$$

Differentiating both sides of series (8) produces a telescopic series which proves the formula since it is true for x = a. This formula, however, is not nearly as useful as the Taylor series because it involves functions which are not necessarily polynomials. But it can be used to derive the appropriate Taylor series for a number of functions. If the remainder term is used, series (7) can be used for an exact integral for ln x, arctan x, and many other functions. Otherwise, series (6) can be used to approximate integrals with two advantages over formulas like Simpson's rule: no commitment to the number of terms is necessary and the error is easily estimated.

This series can also be generalized to include the theory of finite differences but in order to do this it is necessary to define a backward difference operation ∇_h^n such that

$$\nabla_h^1 f(x) = \nabla_h f(x) = \lim \frac{f(x) - f(x-h)}{h}$$

and

$$\nabla_h^n f(x) = \nabla_h \left(\nabla_h^{n-1} f(x) \right)$$

where, once again, h can have any real value including zero, which would provide an alternative definition of the derivative. The backward difference of $\binom{x-a}{r}_h$ is fairly simple:

$$\nabla_h^n \binom{x-a}{r}_h = \binom{x-a-nh}{r-n}_h$$

but is of no apparent use in constructing a series. Curiously, if h = 1 and (x - a) is an integer, successive forward differences follow a right diagonal and successive backward differences follow a left diagonal in Pascal's Triangle. I offer without proof, as a generalization of series (8),

$$f(x) = f(a) + \binom{x-a}{1}_h \nabla_h f(x) - \binom{x-a}{2}_h \nabla_h^2 f(x) + \cdots \qquad (9)$$

where series (8) is a special case where h = 0. I could not use $\Delta_h^n f(x)$ in Equation (9) because the rule for finding the difference of a product is different from the rule for a derivative of a product if $h \neq 0$:

$$\Delta_h(uv) = u\,\Delta_h v + v\,\Delta_h u + h\,\Delta_h u\,\Delta_h v$$
$$\nabla_h(uv) = u\,\nabla_h v + v\,\nabla_h u + h\,\nabla_h u\,\nabla_h v$$
$$d(uv) = u\,dv + v\,du$$

The notation is awkward, but it conforms to the standard mathematical notation if h is omitted in subscripts when equal to one.

I have succeeded in unifying the theories of infinitesimal calculus and the calculus of finite differences to a large extent, as they should be.

VIII. CONCLUSION

In this paper, I discussed some of the many applications of Pascal's Triangle, and how these applications branch out into various fields of mathematics. I have shown how a simple triangle can have significant effects on many practical and interesting theories in mathematics, such as the binomial theorem, combinatorial analysis, Fibonacci numbers, and the calculus of finite differences. It is important to realize that much of mathematics depends on the ability to break a complicated idea down to a simple law.

FOOTNOTES

1. Huntley, p. 6.
2. Beckmann, p. 117
3. Hogben, p. 327.
4. Needham, p. 135.
5. Dudeney, p. 167.
6. Hogben, p. 327.
7. Feller, p. 38.
8. Gardner, December, 1966.
9. *Am. Math. Monthly,* October 1946, p. 236.
10. *Ibid.,* p. 237.
11. *Ibid.*
12. Meyer, p. 69.
13. *Am. Math, Monthly*, February 1940, p. 88.
14. Gardner, August 1959.
15. Huntley, p. 23.
16. Licks, p. 107.
17. *Ibid.,* p. 108.
18. Gardner, August 1959.
19. Vorobyov, pp. 12-15.
20. Huntley, pp. 33-34.
21. Langer, p. 558.

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An excellent exposition of a difficult arithmetic concept
He builds from quite simple concepts to an excellent and nearly original
conclusion with a quite readable exposition.
This should be given consideration for the thesis prize.

$\boxed{A}$

He has obviously done a lot of research and put it all
together to come up with some original ideas.
The typographical errors are few. Very good for a high school
student.

M.P.K.

He includes some entertaining asides from
the history of mathematics.