

FORCING AND NILPOTENT GROUPS

by

Roger Schlafly

EE Senior  
Independent Work  
May, 1976

## §0 Introduction

This paper is concerned with nilpotent groups of class  $\leq n$  for some fixed  $n \geq 2$ . In particular, we investigate the generic models that arise from finite and infinite forcing.

The notation used is taken from Shoenfield and Hirschfeld & Wheeler. In addition, the usual group theoretical conventions will be made, such as writing exponents for repeated multiplication. Also, it is assumed that no variables become bound in substitution.

The chief results of this paper are that finitely generic nilpotent groups are periodic, and that the theory of first order number theory is Turing reducible to infinite forcing companion for nilpotent groups of class  $\leq n$  (for  $n \geq 2$ ). These results are not the best possible, as discussed in § 3 and § 4. Many of the ideas come from Saracino, who shows the following results:

- (1) There is a periodic finitely generic nilpotent group of class  $n$  for each  $n$ .
- (2) He finds the formula  $\text{Inf}_{n-1}$ .
- (3)  $Z(G)$  is periodic if  $G$  is finitely generic.
- (4) The theory of nilpotent groups of class  $\leq n$  has no model companion.

I wish to thank Saracino for sending me reprints and Charles F. Miller, my adviser, for suggesting the topic and making several helpful suggestions.

## §1 Nilpotent Groups

Let  $L$  be the language with symbols  $\circ$ ,  $1$ , and  $^{-1}$ . Let the commutator  $[a, b]$  be an abbreviation for  $a^{-1} \circ b^{-1} \circ a \circ b$ . Iterated commutators  $[x_1, \dots, x_m]$  are defined by induction on  $m$ :

$$[x_1] = x_1$$

$$[x_1, \dots, x_{m+1}] = [[x_1, \dots, x_m], x_{m+1}]$$

Let  $T_n$  ( $n \geq 1$ ) be the (first-order) theory with language  $L$  and axioms:

$$x \circ 1 = x$$

$$x \circ x^{-1} = 1$$

$$(x \circ y) \circ z = x \circ (y \circ z)$$

$$[x_1, \dots, x_{n+1}] = 1$$

The models of  $T_n$  are called nilpotent groups of class  $\leq n$ .

The direct product of models of  $T_n$  is a model of  $T_n$ , so  $T_n$  has the joint embedding property.

Let  $G$  be a nilpotent group of class  $n$ . The lower central series of  $G$  is defined as follows:

$$G^k = [G, \dots, G] \quad (\text{k terms})$$

i.e.,  $G^k$  is the subgroup of  $G$  generated by  $\{[x_1, \dots, x_k] \mid x_1, \dots, x_k \in G\}$ . The upper central series is defined by

$$G_k = \{x \in G \mid \forall z_1, \dots, z_{n-k+1} \in G \quad [x, z_1, \dots, z_{n-k+1}] = 1\}$$

It is not immediate that  $G_k$  is a group.

Lemma 1 If  $a \in G_k$ , then

$$[xa^{-1}y, z_1, \dots, z_{n-k+1}] = [xy, z_1, \dots, z_{n-k+1}]$$

Proof In other words, if  $m = n-k+1$ , this says that if  $a \in G_{n-m+1}$ , then

$$[xa^{\pm 1}y, z_1, \dots, z_m] = [xy, z_1, \dots, z_m]$$

This can be proved by induction on  $m$ . If  $m = 0$ , then

$a \in G_{n+1} = \{1\}$ , so  $xa^{\pm 1}y = xy$ . Assume the above holds for  $m$ . Then for  $a \in G_{n-(m+1)+1} = G_{n-m}$ , and  $x, y, z_1, \dots, z_{m+1} \in G$ , let

$$b = [a, x^{-1}z_1x]$$

Then  $b \in G_{n-m+1}$  and

$$a^{-1}x^{-1}z_1^{-1}x = bx^{-1}z_1^{-1}xa^{-1}$$

Thus

$$\begin{aligned} [xay, z_1, \dots, z_{m+1}] &= [y^{-1}a^{-1}x^{-1}z_1^{-1}xayz, z_2, \dots, z_{m+1}] \\ &= [y^{-1}(bx^{-1}z_1^{-1}xa^{-1})ayz_1, z_2, \dots, z_{m+1}] \\ &= [y^{-1}bx^{-1}z_1^{-1}xyz_1, z_2, \dots, z_{m+1}] \\ &= [y^{-1}x^{-1}z_1^{-1}xyz_1, z_2, \dots, z_{m+1}] \\ &= [xy, z_1, \dots, z_{m+1}] \end{aligned}$$

Also,

$$\begin{aligned} [xa^{-1}y, z_1, \dots, z_{m+1}] &= [y^{-1}ax^{-1}z_1^{-1}xa^{-1}yz_1, z_2, \dots, z_{m+1}] \\ &= [y^{-1}ab^{-1}a^{-1}x^{-1}z_1^{-1}xyz_1, z_2, \dots, z_{m+1}] \\ &= [y^{-1}aa^{-1}x^{-1}z_1^{-1}xyz_1, z_2, \dots, z_{m+1}] \\ &= [xy, z_1, \dots, z_{m+1}] \end{aligned}$$

This completes the induction. //

### Theorem 2

- (i)  $G^k$  and  $G_k$  are normal subgroups of  $G$
- (ii)  $G = G^1 \supseteq G^2 \supseteq \dots \supseteq G^{n+1} = \{1\}$
- (iii)  $G = G_1 \supseteq G_2 \supseteq \dots \supseteq G_{n+1} = \{1\}$
- (iv)  $G^k \subseteq G_k$
- (v)  $G_{n+1} = Z(G)$ , the center of  $G$
- (vi)  $[G_k, G] \subseteq G_{k+1}$
- (vii)  $[G^k, G] = G^{k+1}$

Immediate, except for (vii), which is postponed to Theorem 7(iv). //

To show that the above definition of the upper central series coincides with the usual one, note that for any  $x \in G$ ,  
 $k = 1, 2, \dots, n$

$$\begin{aligned} x \in G_k &\text{ iff } \forall z_1, \dots, z_{n-k+1} [x, z_1, \dots, z_{n-k+1}] = 1 \\ &\text{ iff } \forall z_1 [x, z_1] \in G_{k+1} \\ &\text{ iff } \forall z_1 xz_1 = z_1 x \text{ modulo } G_{k+1} \\ &\text{ iff } xG_{k+1} \in Z(G/G_{k+1}) \end{aligned}$$

This last line is the usual definition.

Let  $y^x$  be defined as  $x^{-1}yx$ . The following easily verified identities are extremely useful.

Theorem 3

- (i)  $[y, x] = [x, y]^{-1} = [x^{-1}, y]^x$
- (ii)  $[x, y]^z = [x^z, y^z]$
- (iii)  $[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$
- (iv)  $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$
- (v)  $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$

Theorem 4 If  $a \in G_k$ , and if  $b, z_1, \dots, z_{n-k} \in G$ , then

$$[ab, z_1, \dots, z_{n-k}] = [az_1, \dots, z_{n-k}][b, z_1, \dots, z_{n-k}]$$

Proof If  $m = n-k$ , the theorem can be restated as:

if  $a \in G_{n-m}$ , then

$$[ab, z_1, \dots, z_m] = [a, z_1, \dots, z_m][b, z_1, \dots, z_m]$$

The proof is by induction on  $m$ . If  $m = 0$ , this is trivial, so assume that it is true for  $m$  and that  $a \in G_{n-(m+1)} = G_{n-m-1}$ ,  
 $b, z_1, \dots, z_{m+1} \in G$ .

Let

$$c = [[a, z_1]^{-1}, b]^{-1}$$

so  $c \in G_{n-m+1}$  and

$$b^{-1}[a, z_1] = c[a, z_1]b^{-1}$$

then

$$\begin{aligned} [ab, z_1, \dots, z_{m+1}] &= [[ab, z_1], z_2, \dots, z_{m+1}] \\ &= [(b^{-1}[a, z_1]b[b, z_1]), z_2, \dots, z_{m+1}] \quad \text{by Theorem 3} \\ &= [(c[a, z_1]b^{-1}b[b, z_1]), z_2, \dots, z_{m+1}] \\ &= [([a, z_1][b, z_1]), z_2, \dots, z_{m+1}] \quad \text{by Lemma 1} \\ &= [a, z_1, \dots, z_{m+1}][b, z_1, \dots, z_{m+1}] \quad \text{by induction hypothesis} \end{aligned}$$

because  $[a, z_1] \in G_{n-m}$ . This completes the induction. //

Corollary 5 If  $a \in G_k$  and  $m \in \mathbb{Z}$ , then

$$[a^m, z_1, \dots, z_{n-k}] = [a, z_1, \dots, z_{n-k}]^m$$

Corollary 6

$$[z_1, \dots, z_n]^m = [[z_1, \dots, z_k]^m, z_{k+1}, \dots, z_n]$$

Let  $A$ ,  $B$ , and  $C$  be normal subgroups of  $G$ . Let  $[A, B, C]$  be the group generated by  $\{(a, b, c) \mid a \in A, b \in B, c \in C\}$ . For any  $x \in G$ ,  $[a, b, c]^x = [a^x, b^x, c^x]$ , so  $[A, B, C]$  is normal in  $G$ .

Theorem 7

- (i)  $[A, B, ] = [B, A]$
- (ii)  $[A, B, C] = [B, A, C]$
- (iii)  $[A, B, C] \subseteq [B, C, A][C, A, B]$
- (iv)  $[[A, B], C] = [A, B, C]$
- (v)  $G^{k+1} = [G^k, G]$

Proof Part (ii) follows from the identity

$$\begin{aligned}[a,b,c]^{-1} &= [[a,b]^{-1}, c][a,b] \\ &= [b,a,c][a,b]\end{aligned}$$

and the fact that for subgroups K and N, with N normal,  
 $KN = NK$  is the group generated by  $KUN$ . Part (iii) follows  
from Theorem 3(v) and similar reasoning. For part (iv),  
it is clear that  $[A,B,C] \subseteq [[A,B],C]$ . The reverse inclusion  
follows from part (ii) and the identity

$$[[a,b],x,c] = [a,b,c]^x[x,c]$$

Part (v) is similar. //

### Theorem 8

$$(i) [G^k, G_{n-k+1}] = 1$$

$$(ii) [G^k, G_{n-k}, G] = 1$$

Proof Both parts can be proved simultaneously, by induction on  $k$ . If  $k = 1$ ,  $[G^k, G_{n-k+1}] = [G, Z(G)] = 1$  and  $[G^k, G_{n-k}, G] = [G, G_{n-1}, G] = [G_{n-1}, G, G] = 1$ . Suppose the theorem is true for  $k$ . Then

$$\begin{aligned}[G^{k+1}, G_{n-k}] &= [[G^k, G], G_{n-k}] \\ &= [G^k, G, G_{n-k}] \\ &\leq [G, G_{n-k}, G^k][G_{n-k}, G^k, G] \\ &= [G_{n-k}, G, G^k][G^k, G_{n-k}, G] \\ &= [[G_{n-k}, G], G^k] \\ &\subseteq [G_{n-k+1}, G^k] \\ &= 1\end{aligned}$$

Also,

$$\begin{aligned}[G^k, G_{n-k}, G] &\leq [G_{n-k}, G, G^k][G, G^k, G_{n-k}] \\ &= [[G_{n-k}, G], G^k][[G^k, G], G_{n-k}] \\ &\subseteq [G_{n-k+1}, G^k][G^{k+1}, G_{n-k}] \\ &= 1 //\end{aligned}$$

## 2 Existentially Complete Nilpotent Groups

For this section, let  $G$  be an existentially complete nilpotent group of class  $n$ . Let  $\text{Inf}_k(b)$  be the formula  $(1 \leq k < n) \forall y (\forall x ([x,y] = 1) \rightarrow \exists z_1, \dots, z_{n-k} (y = [b, z_1, \dots, z_{n-k}]))$ .

Theorem 1 If  $b \in G_k$ , then  $b$  has infinite order modulo  $G_{k+1}$  iff  $G \models \text{Inf}_k(b)$ .

Proof Suppose  $b \in G_k$  has infinite order modulo  $G_{k+1}$ . Let

$$\begin{aligned} H &= G * \langle x_1, \dots, x_{n-k} \rangle \\ H_1 &= H/H^{n+1} \end{aligned}$$

Note that any map from  $\{x_1, \dots, x_{n-k}\}$  to  $G$  extends to a homomorphism from  $H$  to  $G$ . This homomorphism is the identity on  $H^{n+1}$ , so it induces a homomorphism from  $H_1$  to  $G$ . By considering the homomorphism determined by  $x_i \mapsto 1$ , we see that  $G \cap H^{n+1} = \{1\}$ . Thus  $G$  can be considered a subgroup of  $H_1$ .

Suppose

$$a \in G \cap \langle y^{-1}[b, x_1, \dots, x_{n-k}] \rangle$$

in  $H_1$ , where  $y \in Z(G)$ . Note that  $[b, x_1, \dots, x_{n-k}] \in Z(G)$ .

For some  $m \in \mathbb{Z}$ ,

$$a = y^{-m}[b, x_1, \dots, x_{n-k}]^m \in G$$

Since the map  $x_i \mapsto 1$  induces the identity homomorphism on  $G$ ,

$$\begin{aligned} a &= y^{-m}[b, 1, \dots, 1]^m \\ &= y^{-m} \end{aligned}$$

Thus

$$[b, x_1, \dots, x_{n-k}]^m = 1$$

or,  $[b^m, x_1, \dots, x_{n-k}] = 1$

By the homomorphism induced by  $x_i \mapsto g_i$  ( $g_i \in G$ ), it follows that  $\forall g_1, \dots, g_{n-k} \in G$   $[b^m, g_1, \dots, g_{n-k}] = 1$ .

But  $b$  has infinite order modulo  $G_{k+1}$ , so  $m = 0$  and  $a = 1$ .

Thus  $G \cap \langle y^{-1}[b, x_1, \dots, x_{n-k}] \rangle = \{1\}$ .

Let

$$H_2 = G / \langle y^{-1}[b, x_1, \dots, x_{n-k}] \rangle$$

Then  $G$  is embedded in  $H_2$  and

$$H_2 \models \exists z_1, \dots, z_{n-k} \quad y = [b, z_1, \dots, z_{n-k}]$$

so

$$G \models \exists z_1, \dots, z_{n-k} \quad y = [b, z_1, \dots, z_{n-k}]$$

Conversely, suppose  $b^m \in G_{k+1}$ ,  $b \in G_k$  where  $m \neq 0$ .

Then if  $G \models \text{Inf}_k(b)$ , then

$$G \models \forall y \in Z(G) \quad \exists z_1, \dots, z_{n-k} \quad y = [b, z_1, \dots, z_{n-k}]$$

$$G \models \forall y \in Z(G) \quad \exists z_1, \dots, z_{n-k} \quad y^m = [b, z_1, \dots, z_{n-k}]^m$$

$$G \models \forall y \in Z(G) \quad \exists z_1, \dots, z_{n-k} \quad y^m = [b^m, z_1, \dots, z_{n-k}]$$

$$G \models \forall y \in Z(G) \quad \exists z_1, \dots, z_{n-k} \quad y^m = 1$$

$$G \models \forall y \in Z(G) \quad y^m = 1$$

But this is a contradiction because  $G$  clearly has an extension which satisfies

$$\exists x_1, \dots, x_n \quad ([x_1, \dots, x_n]^m \neq 1) \quad //$$

Proposition 2  $G^n = Z(G)$ . In fact, every element of  $Z(G)$  is an  $n$ -fold commutator.

Proof Let  $H$  be a nilpotent group of class  $n$  with an  $(n-1)$ -fold commutator  $b$  of infinite order modulo  $Z(H)$ . Suppose  $a \in Z(G)$ . Then  $a \in Z(G \times H)$ ,  $b \in (G \times H)_{n-1}$  and  $b$  has infinite order modulo  $Z(G \times H) = (G \times H)_n$ . By the proof of Theorem 1,  $G \times H$  has an extension  $K$  such that

$$K \models \exists z_1 (a = [b, z_1])$$

Since  $b$  is an  $(n-1)$ -fold commutator,

$$K \models \exists z_1, \dots, z_n (a = [z_1, \dots, z_n])$$

But  $G$  is existentially complete, so

$$G \models \exists z_1, \dots, z_n (a = [z_1, \dots, z_n])$$

and thus  $a$  is an  $n$ -fold commutator. //

Proposition 3  $Z(G)$  is an existentially complete abelian group.

Proof It is necessary to show that  $Z(G)$  is divisible, and contains infinitely many elements of all orders. To show that  $Z(G)$  has an element of order  $m$ , let  $H$  be a nilpotent group of class  $n$  which contains an  $n$ -fold commutator of order  $m$ . Then

$$\begin{aligned} G \times H \models & \exists x_1, \dots, x_n ([x_1, \dots, x_n] \neq 1 \\ & \& [x_1, \dots, x_n]^2 \neq 1 \& \dots \& \\ & [x_1, \dots, x_n]^{m-1} \neq 1 \& [x_1, \dots, x_n]^m = 1) \end{aligned}$$

so  $G$  also contains such an element. To show divisibility, suppose  $a \in Z(G)$  has infinite order. Let  $H$  be the direct product of  $G$  with  $\mathbb{Z}$ , with  $\langle a \rangle$  amalgamated with  $m\mathbb{Z}$ . Then

$$H \models \exists x (x^m = a)$$

so  $G$  also has  $m$ th roots of  $a$ , for all  $m$ . If  $a$  has finite order, the proof is similar. //

If  $1 \leq k < n$ , let  $\text{Pow}_k(b, c)$  be the formula

$$\begin{aligned} \bigvee & x_1, \dots, x_{n-k+1} ([c, x_1, \dots, x_{n-k+1}] = 1 \\ & \& ([b, x_1, \dots, x_{n-k}] = 1 \\ & \longrightarrow [c, x_1, \dots, x_{n-k}] = 1)) \end{aligned}$$

Let  $\text{Pow}_n(b, c)$  be the formula

$$\exists y_1, \dots, y_n \exists z (b = [y_1, \dots, y_n] \& \text{Pow}_1(y_1, z) \\ \& c = [z, y_2, \dots, y_n])$$

This definition is motivated by the following theorem:

Theorem 4 If  $b \in G_k$ ,  $c \in G$  then  $G \models \text{Pow}_k(b, c)$  iff there is an integer  $m$  such that  $c = b^m$  modulo  $G_{k+1}$ .

Proof Consider first the case where  $1 \leq k < n$ . Suppose  $c = b^m$  modulo  $G_{k+1}$ . For any  $x_1, \dots, x_{n-k+1} \in G$ ,

$$[c, x_1, \dots, x_{n-k+1}] = [b^m(b^{-m}c), x_1, \dots, x_{n-k+1}] \\ = [b^m, x_1, \dots, x_{n-k+1}] \\ = 1$$

and if  $[b, x_1, \dots, x_{n-k}] = 1$  then

$$[c, x_1, \dots, x_{n-k}] = [b^m(b^{-m}c), x_1, \dots, x_{n-k}] \\ = [b^m, x_1, \dots, x_{n-k}] \\ = [b, x_1, \dots, x_{n-k}]^m \\ = 1$$

Conversely, suppose  $G \models \text{Pow}_k(b, c)$  and for each integer  $m$ ,  $c \neq b^m$  modulo  $G_{k+1}$ , let  $l$  be the least positive integer such that there is an integer  $\alpha$  with

$$b^\alpha = c^l \text{ modulo } G_{k+1}$$

Let  $a$  be an element of the center of order  $l$ . If  $l = 1$ , we are done; if no such  $l$  exists, let  $a \in Z(G)$  be any element other than 1. Let  $\langle u_1, \dots, u_{n-k} \rangle$  be the free group on  $u_1, \dots, u_{n-k}$ . Let

$$H_1 = G * \langle u_1, \dots, u_{n-k} \rangle$$

$$H = H_1 / H_1^{n+1}$$

$$K = \langle [b, u_1, \dots, u_{n-k}], a^{-1}[c, u_1, \dots, u_{n-k}] \rangle \subseteq H$$

Suppose

$$x \in K \cap G$$

Since  $K$  is central in  $H$ , there are integers  $\alpha, \beta$  such that

$$x = [b, u_1, \dots, u_{n-k}]^\alpha a^{-\beta} [c, u_1, \dots, u_{n-k}]^\beta$$

The homomorphism determined by  $u_i \mapsto 1$  is the identity on  $G$ , so

$$\begin{aligned} x &= [b, u_1, \dots, u_{n-k}]^\alpha a^{-\beta} [c, u_1, \dots, u_{n-k}]^\beta \\ &= a^{-\beta} [b, u_1, \dots, u_{n-k}]^\alpha [c, u_1, \dots, u_{n-k}]^\beta \\ &= a^{-\beta} [b^\alpha, u_1, \dots, u_{n-k}] [c^\beta, u_1, \dots, u_{n-k}] \\ &= a^{-\beta} [b^\alpha c^\beta, u_1, \dots, u_{n-k}] \\ &= a^{-\beta} [b^\alpha c^\beta, 1, \dots, 1] \\ &= a^{-\beta} \end{aligned}$$

Thus  $[b^\alpha c^\beta, u_1, \dots, u_{n-k}] = 1$ . By the homomorphism determined by  $u_i \mapsto g_i \in G$ , we have  $b^\alpha c^\beta \in G_{k+1}$ . It follows that 1 divides  $\beta$ ; so  $x = a^{-\beta} = 1$ . Hence  $G \cap K = \{1\}$  and  $G$  can be considered to be embedded in  $H/K$ . Since

$$H/K \models \exists u_1, \dots, u_{n-k} ([b, u_1, \dots, u_{n-k}] = 1 \& [c, u_1, \dots, u_{n-k}] \neq 1)$$

it follows that

$$G \models \exists u_1, \dots, u_{n-k} ([b, u_1, \dots, u_{n-k}] = 1 \& [c, u_1, \dots, u_{n-k}] \neq 1)$$

which contradicts  $G \models \text{Pow}_K(b, c)$ .

For the case where  $k = n$ , suppose  $b \in Z(G)$  and  $c = b^m$ .

By Theorem 2, for some  $y_1, \dots, y_n \in G$ ,  $b = [y_1, \dots, y_n]$ . If  $z = y_1^m$  then we clearly have  $G \models \text{Pow}_1(y_1, z)$  and

$$\begin{aligned} c &= b^m \\ &= [y_1, \dots, y_n]^m \\ &= [y_1^m, y_2, \dots, y_n] \\ &= [z, y_2, \dots, y_n] \end{aligned}$$

Conversely, suppose  $G \models \text{Pow}_n(b, c)$ . Then for some  $m \in \mathbb{Z}$ ,  $a_1, \dots, a_n \in G$ ,  $d \in G_2$  we have

$$b = [a_1, \dots, a_n]$$

$$c = [a_1^m d, a_2, \dots, a_n]$$

so

$$c = [a_1^m, a_2, \dots, a_n]$$

$$= [a_1, \dots, a_n]^m$$

$$= b^m //$$

Note that  $\text{Pow}_k(b, c)$  for  $k < n$  does not exactly characterize  $c$  being a power of  $b$ , since we could have  $G \models \text{Pow}_k(b, c)$  without  $b$  necessarily commuting with  $c$ . Let  $\text{Div}(u, v)$  be the formula

$$\exists x_1, \dots, x_n \exists y \forall z (u = [x_1, \dots, x_n] \& v = [x_1, \dots, x_{n-1}, y] \\ \& ((\text{Pow}_{n-1}([x_1, \dots, x_{n-1}], z) \& [z, y] = 1) \rightarrow [z, x_n] = 1))$$

Theorem 5 If  $u, v \in Z(G)$  have finite order, then  $G \models \text{Div}(u, v)$  iff the order of  $u$  divides the order of  $v$ .

Proof Let  $k$  be the order of  $u$  and let  $l$  be the order of  $v$ . Let  $H$  be a nilpotent group of class  $n$  with an  $(n-1)$ -fold commutator  $[a_1, \dots, a_{n-1}]$  of infinite order modulo  $Z(H)$ . Let  $K$  be an existential completion of  $G \times H$ . Then  $[a_1, \dots, a_{n-1}]$  has infinite order modulo  $Z(K)$ . Also  $u, v \in Z(K)$ , because  $u$  and  $v$  are both  $n$ -fold commutators by Theorem 2. Thus  $K \not\models \text{Inf}_{n-1}([a_1, \dots, a_{n-1}])$ , so for some  $u_1, v_1 \in G$ ,

$$u = [a_1, \dots, a_{n-1}, u_1]$$

$$v = [a_1, \dots, a_{n-1}, v_1]$$

and hence

$$K \models \exists x_1, \dots, x_n \exists y (u = [x_1, \dots, x_n] \& v = [x_1, \dots, x_{n-1}, y])$$

$$G \not\models \exists x_1, \dots, x_n \exists y (u = [x_1, \dots, x_n] \& v = [x_1, \dots, x_{n-1}, y])$$

Suppose  $k$  divides  $l$ . Let  $x_1, \dots, x_n, y$  be elements of  $G$  such that

$$u = [x_1, \dots, x_n]$$

$$v = [x_1, \dots, x_{n-1}, y]$$

For any  $z \in G$ , if  $[z, y] = 1$  and

$$z = [x_1, \dots, x_{n-1}]^m \mod Z(G)$$

then

$$\begin{aligned} v^m &= [x_1, \dots, x_{n-1}, y]^m \\ &= [[x_1, \dots, x_{n-1}]^m, y] \\ &= [z, y] \\ &= 1 \end{aligned}$$

Thus  $1 \mid m$  and

$$\begin{aligned} [z, x_n] &= [[x_1, \dots, x_{n-1}]^m, x_n] \\ &= [x_1, \dots, x_n]^m \\ &= u^m \end{aligned}$$

But  $k \nmid m$ , so  $u^m = 1$ . Therefore,

$$G \models \text{Div}(u, v)$$

Conversely, suppose  $G \models \text{Div}(u, v)$ . Let  $x_1, \dots, x_n, y \in G$  be such that  $u = [x_1, \dots, x_n]$ ,  $v = [x_1, \dots, x_{n-1}, y]$ , and  $G \models \forall z \ (\text{Pow}_{n-1}([x_1, \dots, x_{n-1}], z) \ \& \ [z, y] = 1 \rightarrow [z, x_n] = 1)$ .

In particular, let  $z = [x_1, \dots, x_{n-1}]^l$  so

$$\begin{aligned} [z, y] &= [[x_1, \dots, x_{n-1}]^l, y] \\ &= [x_1, \dots, x_{n-1}, y]^l \\ &= v^l \\ &= 1 \end{aligned}$$

so by hypothesis,

$$\begin{aligned}
 1 &= [z, x_n] \\
 &= [[x_1, \dots, x_{n-1}]^1, x_n] \\
 &= [x_1, \dots, x_n]^1 \\
 &= u^1
 \end{aligned}$$

and thus  $k$  divides  $l$ . //

Let  $\text{Period}_k$  be the formula  $\forall u ((\forall z_1, \dots, z_{n-k+1}$   
 $[u, z_1, \dots, z_{n-k+1}] = 1) \rightarrow \neg \text{Inf}_k(u))$ .

Proposition 6  $G \models \text{Period}_k$  iff  $G_k/G_{k+1}$  is periodic.

Proof The formula  $\text{Period}_k$  is equivalent to

$$\forall u (u \in G_k \rightarrow \neg \text{Inf}_k(u))$$

The result follows from Theorem 1. //

### § 3 Finitely Generic Nilpotent Groups

We consider finite forcing for the theory of nilpotent groups of class  $n$  in this section.

Lemma 1 If  $p$  is a finite condition, then  $p$  has a finite model.

Proof Let  $c_1, \dots, c_m$  be the constants occurring in  $p$ . Let  $G$  be a model of  $T_n \cup p$ , and let  $H$  be the subgroup of  $G$  generated by  $\{c_1, \dots, c_m\}$ . By a theorem of P. Hall, finitely generated nilpotent groups are residually finite, it follows that  $H$  has a finite quotient group that satisfies  $p$ . //

Proposition 2 There exists a periodic finitely generic nilpotent group  $H$  of class  $n$ .

Proof Let  $C = \{c_0, c_1, c_2, \dots\}$  be a countable set of constant symbols, and let  $A_1, A_2, \dots$  be an enumeration of all sentences of  $L(C)$ . Inductively, define a sequence  $\{p_n\}$  of conditions as follows. Let  $p_1 = \emptyset$ . Suppose  $p_{2n-1}$  has been defined. If  $p_{2n-1}$  forces  $\neg A_n$ , let  $p_{2n} = p_{2n-1}$ . Otherwise, take some  $p_{2n}$  containing  $p_{2n-1}$  such that  $p_{2n}$  forces  $A_n$ . By Theorem 1, for some  $m$ ,  $p_{2n}$  has a model of order  $m$ . Let  $c_1, \dots, c_k$  include the constant symbols occurring in  $p_{2n}$ . Let

$$p_{2n+1} = p_{2n} \cup \{c_1^m = 1, \dots, c_k^m = 1\}$$

which is consistent, for it has a model of order  $m$ . This defines the conditions  $p_1 \subseteq p_2 \subseteq \dots$ .

The set  $\{q \mid q \subseteq p_n \text{ for some } n\}$  is (finitely) generic, and therefore generates a (finitely) generic model  $H$ . Every

element of  $G$  is named by a constant in  $C$ , and therefore has finite order. Thus  $H$  is periodic. //

Proposition 3  $T_n^f \vdash \exists y_1 \forall x_1 \dots x_n \exists y_2 \dots y_n [x_1, \dots, x_n] = [y_1, \dots, y_n]$

Proof Suppose, to the contrary, that for some  $c_1 \in C$ , and some condition  $p$ ,

$$p \Vdash \forall x_1, \dots, x_n \exists y_2, \dots, y_n [x_1, \dots, x_n] = [c_1, y_2, \dots, y_n]$$

By lemma 1, there is an  $m > 0$  such that

$$p_0 = p \vee \{c_1^m = 1\}$$

is a condition. Let  $d_1, \dots, d_n$  be new constants, and let  $p_1 = p_0 \vee \{[d_1, \dots, d_n]^m \neq 1\}$ . Then there is an extension  $p_2 \supseteq p_1$  such that for some constants  $c_2, \dots, c_n$ ,

$$p_2 \Vdash [d_1, \dots, d_n] = [c_1, \dots, c_n]$$

But, in any model of  $p_2$

$$\begin{aligned} [d_1, \dots, d_n]^m &= [c_1, \dots, c_n]^m \\ &= [c_1^m, c_2, \dots, c_n] \\ &= [1, c_2, \dots, c_n] \\ &= 1 \end{aligned}$$

which is a contradiction. //

Theorem 4 If  $G$  is a finitely generic nilpotent group of class  $n$ ,  $G$  is periodic.

Proof Let  $H$  be the periodic group of Theorem 2. Since  $T_n$  has the joint embedding property,  $T_n^f$  is complete, so  $G$  is elementarily equivalent to  $H$ .

We first show that  $G_k/G_{k+1}$  is periodic for  $1 \leq k < n$ .

By Proposition 2.6,  $H \models \text{Period}_k$ , so  $G \models \text{Period}_k$ . By Proposition 2.6 again,  $G_k/G_{k+1}$  is periodic.

Next we show that  $Z(G) = G_n/G_{n+1}$  is periodic. If  $a \in Z(G)$ , then by Proposition 2.2,  $a$  is of the form  $[a_1, \dots, a_n]$ . Since  $G_{n-1}/G_n$  is periodic, there is an  $m > 0$  such that  $[a_1, \dots, a_{n-1}]^m \in G_n = Z(G)$ . Thus  $a^m = [a_1, \dots, a_n]^m = [[a_1, \dots, a_{n-1}]^m a_n] = 1$  so  $a$  has finite order.

Suppose  $C \in G$ . By induction on  $k$ , we will show that for each  $k$  there is an  $m > 0$  such that  $C^m \in G_k$ . If  $k = 1$ , we can let  $m = 1$ . If  $C^m \in G_k$  ( $m > 0$ ), then since  $G_k/G_{k+1}$  is periodic, there is an  $m' > 0$  such that  $(C^m)^{m'} = C^{mm'} \in G_{k+1}$ . This completes the induction, so there is an  $m > 0$  such that  $C^m \in G_{n+1} = \{1\}$ . Thus  $G$  is periodic. //

Since  $T_n$  is finitely axiomatized, it is known that  $T_n^f$  is Turing reducible to  $\text{Th}(\mathbb{N})$ . There is some reason to believe that  $T_n^f$  is in fact Turing equivalent to  $\text{Th}(\mathbb{N})$ , for  $n > 1$ . In attempting to show this, we have tried to define arithmetic in finitely generic nilpotent groups of class  $n$ . One possible strategy would be to interpret a positive integer  $m$  as an equivalence class of elements of the center that have order  $m$ . This equivalence class is nonempty by Proposition 2.3. The statement "a and b have the same order" is definable by  $\text{Div}(a, b) \wedge \text{Div}(b, a)$ . However, the difficulty with this method lies with defining addition and multiplication. J. Robinson has shown that addition and multiplication are definable in terms of divisibility in terms of divisibility and the successor function, but the author has been unable to define the successor function.

## § 4 Infinitely Generic Nilpotent Groups

For this section, let  $G$  be an infinitely generic nilpotent group of class  $n$ .

Proposition 1  $G \models \exists x_1, \dots, x_k \text{ Inf}_k([x_1, \dots, x_k])$

Proof Since  $T_n$  has the joint embedding property,  $T_n^F$  is complete, so we may take  $G$  to be existentially universal.

The existential type  $\{ \exists x_{k+1}, \dots, x_{n-k} ([x_1, \dots, x_n]^m \neq 1) \mid m = 1, 2, 3, \dots \}$  is consistent, because it is satisfied by any nilpotent group of class  $n$  having an  $n$ -fold commutator of infinite order. By Corollary 1.6 and Theorem 2.1, an existentially complete nilpotent group satisfies  $\exists x_1, \dots, x_k \text{ Inf}_k([x_1, \dots, x_k])$  iff it satisfies the above existential type. Since  $G$  satisfies this type, the proof is complete. //

Let  $\text{Free}(a, b)$  be the formula  $\exists x_1, \dots, x_{n-1} (a = [x_1, \dots, x_{n-1}] \& \text{Inf}_{n-1}(a) \& \forall y (\text{Pow}_{n-1}(a, y) \rightarrow \forall z ([y, z] = 1) \vee [y, b] \neq 1))$ . For the rest of this section, let  $a, b$ , and  $c$  be elements of  $G$  such that  $G \models \text{Free}(a, b) \& c = [a, b]$ . This assumption is justified by the next proposition.

Proposition 2  $G \models \exists a \exists b \text{ Free}(a, b)$

Proof As before, assume  $G$  is existentially universal.

By Proposition 1,  $G$  has an  $(n-1)$ -fold commutator  $a$  which has infinite order modulo the center of  $G$ .  $G$  satisfies the existential type  $\{ [x_1, \dots, x_n]^1 \neq 1 \& \dots \& [x_1, \dots, x_m]^m \neq 1 \mid m = 1, 2, 3, \dots \}$  so  $Z(G)$  has an element  $c$  of infinite order. By Theorem 2.1, there is a  $b$  in  $G$  such that  $[a, b] = c$ . If  $y = a^m$  modulo  $Z(G)$ , then either  $m = 0$ , in which case  $y \in Z(G)$ , or  $m \neq 0$ , in which

case  $[y, b] = [a^m, b] = [a, b]^m = c^m \neq 1$ . Thus  $G \models \text{Free}(a, b)$ . //

Proposition 3

$$(1) a^{k_l} b^{l_a} a^{k'_l} b^{l'_a} = a^{k+k'_l} b^{l+l'_a} c^{-k'_l l_a}$$

$$(2) [a^{k_l} b^{l_a}, a^{k'_l} b^{l'_a}] = c^{k_l l'_a - k'_l l_a}$$

Proof

$$b^{l_a} a^{k_l} = a^{k_l} b^{l_a} [b^{l_a}, a^{k_l}]$$

$$= a^{k_l} b^{l_a} [a^{k_l}, b^{l_a}]^{-1}$$

$$= a^{k_l} b^{l_a} [a, b]^{-k_l l_a}$$

$$= a^{k_l} b^{l_a} c^{-k_l l_a}$$

$$a^{k_l} b^{l_a} a^{k'_l} b^{l'_a} = a^{k_l} a^{k'_l} b^{l_a} c^{-k'_l l_a} \cdot b^{l'_a}$$

$$= a^{k+k'_l} b^{l+l'_a} c^{-k'_l l_a} \quad \text{since } c \in Z(G)$$

$$[a^{k_l} b^{l_a}, a^{k'_l} b^{l'_a}] = (a^{k'_l} b^{l'_a} a^{k_l})^{-1} (a^{k_l} b^{l_a} a^{k'_l} b^{l'_a})$$

$$= (a^{k'_l+k_l} b^{l'_a+l_a} c^{-k_l l_a})^{-1} (a^{k+k'_l} b^{l+l'_a} c^{-k'_l l_a})$$

$$= c^{k_l l'_a - k'_l l_a} //$$

Proposition 4  $\langle a, b \rangle$  is isomorphic to the free nilpotent group of class 2 and rank 2.

Proof Since  $G \models \text{Inf}_{n=1}^\infty(a)$ ,  $a$  has infinite order.

Also, for any  $m \neq 0$ ,  $[a^m, b] \neq 1$ , so  $[a, b]^m = [a, b]^m \neq 0$  and

hence  $b$  and  $[a, b]$  have infinite order. If  $c = [a, b]$ , then every element of  $\langle a, b \rangle$  can be uniquely written in the form  $a^{k_l} b^{l_a} c^m$ , for if  $a^{k_l} b^{l_a} c^m = a^{k'_l} b^{l'_a} c^{m'}$  then

$$a^{k-k'_l} b^{l-l'_a} c^{m-m'} = 1$$

$$\text{so } 1 = [a, 1]$$

$$= [a, a^{k-k'_l} b^{l-l'_a} c^{m-m'}]$$

$$= [a, a^{k-k'_l} b^{l-l'_a}]$$

$$= c^{l-l'_a}$$

$$1 = 1'$$

Also

$$\begin{aligned}1 &= [1, b] \\&= [a^{k-k'} b^{l-l'} c^{m-m'}, b] \\&= [a^{k-k'} b^{l-l'}, b] \\&= c^{k-k'}\end{aligned}$$

$$k = k^t$$

Finally

$$1 = a^{k-k'} b^{l-l'} c^{m-m'}$$

so  $m = m'$  because  $c$  has infinite order. This completes the proof.  $\langle a, b \rangle$  can be written as a matrix group by the correspondence

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad //$$

Let  $\text{Prod}(a, b; u, v, w)$  be the formula

$$\exists x \exists y (Pow_{n-1}(a, x) \wedge Pow_1(b, y) \wedge [x, b] = u \wedge [a, y] = v \wedge [x, y] = w)$$

Proposition 5.  $G \models \text{Prod}(a; b; c^k; c^l; c^m)$  iff  $k + l = m$

Proof If  $k + l = m$ , let  $x = a^k$  and  $y = b^l$ .

50

$$[x, b] = [x^k, b] = [a, b]^k \in C^k$$

$$[a, y] = [a, b^l] = [a, b]^l = c^l$$

$$[x, y] = [a^k, b^l] = [a, b]^{kl} = c^{kl} = c^m$$

Conversely, suppose

$$G \models \text{Pow}_{\prod_{k=1}^n}(\bar{a}, \bar{s}) \And \text{Pow}_1(\bar{b}, \bar{t}) \And [\bar{s}, \bar{b}] = \bar{c}^k \And [\bar{a}, \bar{t}] = \bar{c}^l \\ \And [\bar{s}, \bar{t}] = \bar{c}^m$$

For some  $k^t, l^t$ ,  $s = a^{k^t} \pmod{G_n}$  and  $t = b^{l^t} \pmod{G_n}$ , so

$$\begin{aligned}
 c^k &= [s, b] \\
 &= [a^{k^t} (a^{-k^t} s), b] \\
 &= [a^{k^t}, b] \quad \text{since } a^{-k^t} s \in Z(G) \\
 &= [a, b]^{k^t} \\
 &= c^{k^t}
 \end{aligned}$$

Hence  $k = k^t$  and

$$\begin{aligned}
 c^l &= [a, t] \\
 &= [a, b^{l^t} (b^{-l^t} t)] \\
 &= [a, b^{-l^t} t] [a, b^{l^t}] \quad \text{by Theorem 1.3} \\
 &= [a, b^{l^t}] \\
 &= [a, b]^{l^t} \\
 &= c^{l^t}
 \end{aligned}$$

Where  $[a, b^{-l^t} t] = 1$  because  $a \in G^{n-1}$ ,  $b^{-l^t} t \in G_2$ , and by Theorem 1.8. Thus  $l = l'$  and

$$\begin{aligned}
 c^m &= [s, t] \\
 &= [a^k (a^{-k} s), t] \\
 &= [a^k, t] \quad a^{-k} s \in Z(G) \\
 &= [a, t]^k \\
 &= (c^l)^k \quad \text{since } [a, t] = c^l \\
 &= c^{lk}
 \end{aligned}$$

so  $m = k + l$ . //

Let  $\text{Sum}(u, v, w)$  be the formula  $u * v = w$ .

Proposition 6  $G \models \text{Sum}(c^k, c^l, c^m)$  iff  $k+l = m$ .

Proof Immediate, since  $c$  has infinite order. //

Let  $N(a, b, c; w)$  be the formula  $\exists u_1, u_2, u_3, u_4$   
 $\exists v_1, v_2, v_3, v_4 \quad (\text{Pow}_n(c, u_1) \& \text{Pow}_n(c, u_2) \& \text{Pow}_n(c, u_3) \&$   
 $\text{Pow}_n(c, u_4) \& \text{Prod}(a, b; u_1, u_1, v_1) \& \text{Prod}(a, b; u_2, u_2, v_2) \&$   
 $\text{Prod}(a, b; u_3, u_3, v_3) \& \text{Prod}(a, b; u_4, u_4, v_4) \& w = v_1 + v_2 + v_3 + v_4$   
 $\& \text{Pow}_n(c, w))$

Proposition 7  $G \models N(a, b, c; w)$  iff  $w = c^k$  for some  $k \geq 0$ .

Proof This follows from a famous theorem of Lagrange that an integer is nonnegative iff it is the sum of four squares. //

Lemma 8 Any atomic formula in the language of the natural numbers (i.e.  $\{0, 1, +, \cdot\}$ ) is equivalent to a formula of the form  $\exists u_1, \dots, u_k (E_1 \& \dots \& E_l)$  where each  $E_i$  has the form  $t_1 = t_2$ ,  $t_1 + t_2 = t_3$ , or  $t_1 \cdot t_2 = t_3$  and each  $t_i$  is 0, 1, or a variable.

Proof By induction on the number of occurrences of  $+$  and  $\cdot$  in the atomic formula, which must be of the form  $P_1 = P_2$ . There are several cases to consider, but a typical case is of the form  $P_1 + P_2 = Q$ . If  $x, y, z$  are variables not occurring in  $P_1 + P_2 = Q$ , then this is equivalent to

$\exists x \exists y \exists z (x = P_1 \& y = P_2 \& z = Q \& x + y = z)$   
and each conjunct contains fewer occurrences of  $+$  and  $\cdot$  than  $P_1 + P_2 = Q$ .

Define a map  $*$  as follows. Let  $0^*$  be 1 and  $1^*$  be  $c$ . If  $x$  is a variable, let  $x^*$  be  $x$ . Suppose  $E_i$  is of the form of Lemma 7. If  $E_i$  is  $t_1 = t_2$ , let  $E_i^*$  be  $t_1^* = t_2^*$ . If  $E_i$  is  $t_1 + t_2 = t_3$ , let  $E_i^*$  be  $\text{Prod}(a, b; t_1^*, t_2^*, t_3^*)$ .

By induction on formulas, \* can be defined on any formula in the language of the natural numbers. If A is atomic, by Lemma 7, A is equivalent to a formula of the form  $\exists u_1, \dots, u_k (E_1 \& \dots \& E_k)$ . Let  $A^*$  be  $\exists u_1, \dots, u_k (N(a, b, c; u_1) \& \dots \& N(a, b, c; u_k) \& E_1^* \& \dots \& E_k^*)$ . Also, let  $(\neg A)^*$  be  $\neg A^*$ , let  $(A \vee B)^*$  be  $A^* \vee B^*$ , and let  $(\exists x A)^*$  be  $\exists x(N(a, b, c; x) \& A^*)$ .

Theorem 9  $\text{Th}(N) \leqslant T_n^F$  ( $n > 1$ )

Proof. From previous results, it is clear that for a closed formula A,  $\mathcal{N} \models A$  iff  $G \models A^*(a, b, c)$

$$\begin{aligned} &\text{iff } G \models \exists a \exists b \exists c (\text{Free}(a, b) \& \\ &\quad c = [a, b] \& A^*(a, b, c)) \end{aligned}$$

$$\begin{aligned} &\text{iff } T_n^F \models \exists a \exists b \exists c (\text{Free}(a, b) \\ &\quad \& c = [a, b] \& A^*(a, b, c)) \end{aligned}$$

This reduction is obviously effective. //

Since  $T_n$  is finitely axiomatized, it is known that  $T_n^F$  must be Turing reducible to second order number theory. One would expect that the degree of  $T_n^F$  is the same as either first or second order number theory. However we have been unable to show this.

Since  $T_n$  has the joint embedding property, it is known (H & W p. 128) that if  $T_n$  has a countable existentially universal structure, then  $T_n^F$  is a  $\Delta_2^1$  set. On the other hand, if it can be shown that existentially universal models of  $T_n$  must have cardinality at least  $2^{\aleph_0}$ , then perhaps a similar argument could be used to interpret second order number theory in these existentially universal models. The author has been umble to settle this question.

## BIBLIOGRAPHY

1. G. Baumslag, Lecture Notes on Nilpotent Groups, American Mathematical Society, Providence, (1971).
2. R. Hall, On the Finiteness of Certain Soluble Groups, Annals of Math (2) 96 (1972), pp. 53-97.
3. J. Hirschfeld and W. Wheeler, Forcing, Arithmetic, Division Rings, Springer-Verlag, Berlin (1975).
4. J. Keisler, Forcing and the Omitting Types Theorem, Studies in Model Theory, Mathematical Association of America, Washington (1973).
5. C. F. Miller, Some Connections Between Hilbert's Tenth Problem and the Theory of Groups.
6. D. Saracino, Wreath Products and Existentially Complete Solvable Groups, Transactions of the American Mathematical Society (1974), pp. 327-339.
7. \_\_\_\_\_, Existentially Complete Nilpotent Groups, to appear.
8. W. R. Scott, Group Theory, Prentice-Hall, Inglewood Cliffs, NJ (1964).
9. J. Shoenfield, Mathematical Logic, Addison-Wiley, Reading (1973).